

PREPARED FOR SUBMISSION TO JCAP

Computation of the Power Spectrum in Chaotic $\frac{1}{4}\lambda\phi^4$ Inflation

Clara Rojas^a and Víctor M. Villalba^b

^aInstituto Venezolano de Investigaciones Científicas (IVIC), Centro de Estudios Interdisciplinarios de la Física, Caracas 1020A, Venezuela.

^bSchool of Mathematical Sciences, Faculty of Science, Monash University, Australia

E-mail: clararoj@gmail.com, Victor.Villalba@monash.edu

Abstract. The phase-integral approximation devised by Fröman and Fröman, is used for computing cosmological perturbations in the quartic chaotic inflationary model. The phase-integral formulas for the scalar power spectrum are explicitly obtained up to fifth order of the phase-integral approximation. As in previous reports [1–3], we point out that the accuracy of the phase-integral approximation compares favorably with the numerical results and those obtained using the slow-roll and uniform approximation methods.

Keywords: Cosmological Perturbations; Chaotic Inflation; Semiclassical Methods.

Contents

1	Introduction	1
2	Equation for the perturbations	2
3	Methodology	5
3.1	Phase-integral approximation	5
3.2	Improved Uniform approximation	7
3.3	Slow-roll approximation	8
3.4	Numerical solution	9
4	Results	10
5	Conclusion	13

1 Introduction

Chaotic inflationary models like $V \propto \phi^p$ have been studied and analyzed with the WMAP data in recent years. The recent observations reported by WMAP favors inflation, and the quartic chaotic model, with $p = 4$ [4, 5] still remains ruled out by this data [6, 7]. Ramirez *et al.* [8] point out that if the scalar power spectrum is considered not scale invariant, then quartic models would not be excluded by the recent cosmological data. Ramirez *et al.* [8] consider a new scenario where the observable modes cross the horizon near the beginning of inflation.

In order to compare with observations, we should be able to obtain very accurate results for the predicted power spectrum of primordial perturbations for a variety of inflationary scenarios. In general, most of the inflationary models are not exactly solvable and approximate or numerical methods are mandatory in the computation of the scalar and power spectra. Traditionally, the method of approximation applied in inflationary cosmology is the slow-roll approximation [9], which produces reliable results in inflationary models with smooth potentials, but cannot be improved on a simple way beyond the leading order. Recently, some authors have applied alternative approximations, such as the WKB method with the Langer modification [10–12], the Green function method [13], and the improved WKB method [14, 15].

Habib *et al* [16–18] have successfully applied the uniform approximation method in the calculation of the scalar and tensor power spectra and the corresponding spectral indices for the quadratic and quartic chaotic inflationary models, showing that the the uniform approximation gives more accurate results than the slow-roll approximation. Casadio *et al* [19] have applied the method of comparison equation to study cosmological perturbations during inflation. The comparison method is based on the uniform approximation proposed by Dingle [20] and Miller [21] and thoroughly discussed by Berry and Mount [22].

Recently [1–3], the phase integral method has been used in the calculation of the scalar and tensor power spectra and the spectral indexes for the power-law and quadratic chaotic model, obtaining better results than those derived using slow-roll [9], WKB and the improved uniform approximation methods [16–18].

Since the 7-year WMAP data does not show evidence for tensor modes [23, 24], in this article we use the phase-integral approximation up to fifth order to calculate the scalar power spectrum for the quartic chaotic inflationary model. We have calculated the tensor power spectrum and since the results are very similar to those obtained in the scalar case, we decided to skip their presentation in this article.

The article is structured as follows: In Sec. 2 we numerically solve the equation governing the scalar perturbations. In Sec. 3 we apply the phase-integral approximation to the quartic chaotic inflationary model. We present the improved uniform approximation method, the slow-roll approximation and the numerical solution to the perturbation equation. The comparison among different methods is presented in Sec. 4. Finally in Sec. 5 we present our conclusions.

2 Equation for the perturbations

In an inflationary universe driven by a scalar field, the equations of motion for the inflaton ϕ and the Hubble parameter H are given by [25]

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V(\phi)}{\partial \phi}, \quad (2.1)$$

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[V(\phi) + \frac{1}{2}\dot{\phi}^2 \right], \quad (2.2)$$

with $V(\phi) = \frac{1}{4}\lambda\phi^4$ and the constant λ , can be fixed with the help of the amplitude of the density fluctuations observed by WMAP. A fitting to the observational data requires that $\lambda \simeq 10^{-13}$ [18]. Using Eq. (2.1) and Eq. (2.2) we obtain that, in the slow-roll approximation, the expansion factor $a(t)$ and the inflaton field $\phi(t)$ are

$$a \simeq a_i \left\{ \frac{\phi_i^4}{8M_{\text{Pl}}^2} \left[1 - \exp \left(-\sqrt{\frac{16\lambda}{3}} M_{\text{Pl}} t \right) \right] \right\}, \quad (2.3)$$

$$\phi(t) \simeq \phi_i \exp \left(-2\sqrt{\frac{\lambda}{3}} M_{\text{Pl}} t \right), \quad (2.4)$$

where a_i and ϕ_i are integration constants corresponding to the initial values of the scale factor and the inflaton.

The application of the the phase-integral approximation for solving the equations governing the scalar perturbations requires the knowledge of the evolution of the expansion factor a and the scalar field ϕ , and the computation of analytic expression of them would highly simplify the calculations. In order to find ϕ and a as a function of time, we solve the coupled system of differential equations (2.1)-(2.2) with the sixth-order Runge-Kutta method [26] and fit the numerical data with the software *gnuplot* following the form of the slow-roll solutions (2.3,2.4). The integration is performed in the physical time t using the sixth-order Runge-Kutta method [26]. The initial value for the inflaton is $\phi_i = 21.9 M_{\text{Pl}}$. The initial condition for the speed of the scalar field $\dot{\phi}_i$ is obtained from the slow-roll solution (2.4). The initial value for a_i is chosen as unity and $\lambda = 1.75 \times 10^{-13}$. These initial conditions guarantee sufficient inflation. We find that the inflation ends at $t_f = 4.62 \times 10^6 M_{\text{Pl}}^{-1} = 1.25 \times 10^{-36} \text{s}$, corresponding to 59.98 e-folds before the scalar field starts to oscillate.

The equation governing the evolution of the scalar perturbations in the quartic chaotic inflationary model is given by

$$\ddot{u}_k + \frac{\dot{a}}{a} \dot{u}_k + \frac{1}{a^2} \left[k^2 - \frac{(\dot{a} \dot{z}_S + a \ddot{z}_S) a}{z_S} \right] u_k = 0. \quad (2.5)$$

In order to apply the phase-integral approximation we need the expressions for $a(t)$ and $\phi(t)$ which have been derived fitting the numerically solution of Eqs. (2.1) and (2.2). We have obtained the following analytic expressions

$$a(t) = a_i \exp \left\{ \bar{\alpha} [1 - \exp(-\bar{\beta}t)] \right\}, \quad (2.6)$$

$$\phi(t) = \phi_i \exp(-\bar{\gamma}t), \quad (2.7)$$

where $\bar{\alpha} = 60.2839$, $\bar{\beta} = 9.6370 \times 10^{-7} M_{\text{Pl}}$, and $\bar{\gamma} = 4.8305 \times 10^{-7} M_{\text{Pl}}$ and, the function $z_S(t)$ has the form

$$z_S(t) = -a_i \phi_i \frac{\bar{\gamma}}{\bar{\alpha} \bar{\beta}} \exp \left\{ \bar{\alpha} [1 - \exp(-\bar{\beta}t)] + \bar{\beta}t - \bar{\gamma}t \right\}. \quad (2.8)$$

Since we want to eliminate the term \dot{u}_k in Eq. (2.5), we introduce the change of variable $u_k(t) = \frac{U_k(t)}{\sqrt{a}}$, obtaining that U_k satisfies the differential equation:

$$\ddot{U}_k + R_S(k, t) U_k = 0, \quad (2.9)$$

with

$$R_S(k, t) = \frac{1}{a^2} \left[k^2 - \frac{(\dot{a} \dot{z}_S + a \ddot{z}_S) a}{z_S} \right] + \frac{1}{4a^2} (a^2 - 2a\ddot{a}), \quad (2.10)$$

where $U(k)$ satisfies the asymptotic conditions

$$U_k \rightarrow A_k \sqrt{a(t)} z_S(t), \quad k t \rightarrow \infty, \quad (2.11)$$

$$U_k \rightarrow \sqrt{\frac{a(t)}{2k}} \exp[-ik\eta(t)], \quad k t \rightarrow 0. \quad (2.12)$$

We now proceed to write the explicit equations for quartic chaotic inflation. From Eq. (2.10), with Eq. (2.6) and Eq. (2.7) we obtain

$$\begin{aligned} R_S(k, t) = & \frac{k^2}{a_i^2} \exp \left\{ -2\bar{\alpha} [1 - \exp(-\bar{\beta}t)] \right\} - \frac{9}{4} \bar{\alpha}^2 \bar{\beta}^2 \exp(-2\bar{\beta}t) \\ & - \frac{3}{2} \bar{\alpha} \bar{\beta} (\bar{\beta} - 2\bar{\gamma}) \exp(-\bar{\beta}t) - (\bar{\beta} - \bar{\gamma})^2. \end{aligned} \quad (2.13)$$

In order to apply the asymptotic condition (2.12), we use the relationship between η and the physical time t , which is given in terms of the integral exponential function $\text{Ei}(z)$ [27] as

$$\eta(t) = \frac{1}{a_i \bar{\beta}} \exp(-\bar{\alpha}) \left\{ \text{Ei}[\bar{\alpha} \exp(-\bar{\beta} t_0)] - \text{Ei}[\bar{\alpha} \exp(-\bar{\beta} t)] \right\}, \quad (2.14)$$

where $t_i = 10^7 M_{\text{Pl}}^{-1}$, so that η is zero at the end of the inflationary epoch. The dependence of η on t is shown in Fig. 1. We can observe that, as

$$-k\eta \rightarrow 0 \Rightarrow kt \rightarrow \infty, \quad (2.15)$$

$$-k\eta \rightarrow \infty \Rightarrow kt \rightarrow 0. \quad (2.16)$$

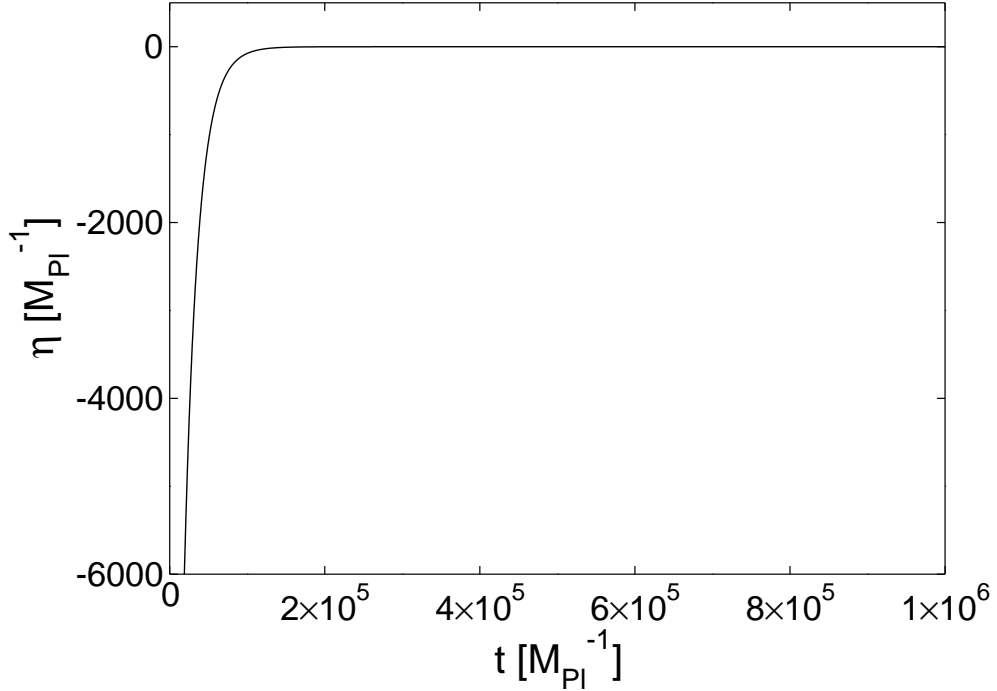


Figure 1. Behavior of η as a function of t for the chaotic inflationary $\frac{1}{4}\lambda\phi^4$ model.

Eq. (2.9), where $R_S(k, t)$ is given by Eq. (2.13), does not possess exact analytic solution. In order to solve the differential equations governing the scalar perturbations in the physical time t , we use the fifth-order phase integral approximation [28] and compare this results with those obtained using the slow-roll and the improved uniform approximation.

Once the mode equation for scalar perturbations is solved for different momenta k , the power spectra for scalar modes is given by the expression

$$P_S(k) = \lim_{-k\eta \rightarrow 0} \frac{k^3}{2\pi^2} \left| \frac{u_k(\eta)}{z(\eta)} \right|^2. \quad (2.17)$$

The spectral indice is defined as

$$n_S(k) = 1 + \frac{d \ln P_S(k)(k)}{d \ln k}. \quad (2.18)$$

3 Methodology

3.1 Phase-integral approximation

In order to solve Eq. (2.9) with the help of the phase-integral approximation [28], we choose the following base functions Q_S for the scalar perturbations

$$Q_S^2(k, t) = R_S(k, t), \quad (3.1)$$

where $R_S(k, t)$ is given by Eq. (2.13). Using this selection, the phase-integral approximation is valid as $kt \rightarrow \infty$, limit where we should impose the condition (2.11), where the validity condition $\mu \ll 1$ holds. The selection, given in Eq. (3.1), makes the first order phase-integral approximation coincides with the WKB solution. The bases functions $Q_S(k, t)$ possess turning points $t_{\text{ret}} = v_S = 117726 M_{\text{Pl}}^{-1}$ for the mode $k = 0.05 \text{ Mpc}^{-1}$. The turning point represents the horizon. There are two ranges where to define the solution. To the left of the turning point $0 < t < t_{\text{ret}}$ we have the classically permitted region $Q_S^2(k, t) > 0$ and to the right of the turning point $t > t_{\text{ret}}$ corresponding to the classically forbidden region $Q_S^2(k, t) < 0$, such as it is shown in Figs 2(a).

The mode k equations for the scalar perturbations (2.9) in the phase-integral approximation has two solutions: For $0 < t < t_{\text{ret}}$

$$\begin{aligned} u_k^{\text{pi}}(t) &= \frac{c_1}{\sqrt{a(t)}} \left| q_S^{-1/2}(k, t) \right| \cos \left[|\omega_S(k, t)| - \frac{\pi}{4} \right] \\ &+ \frac{c_2}{\sqrt{a(t)}} \left| q_S^{-1/2}(k, t) \right| \cos \left[|\omega_S(k, t)| + \frac{\pi}{4} \right]. \end{aligned} \quad (3.2)$$

For $t > t_{\text{ret}}$

$$\begin{aligned} u_k^{\text{pi}}(t) &= \frac{c_1}{2\sqrt{a(t)}} \left| q_S^{-1/2}(k, t) \right| \exp \left[-|\omega_S(k, t)| \right] \\ &+ \frac{c_2}{\sqrt{a(t)}} \left| q_S^{-1/2}(k, t) \right| \exp \left[|\omega_S(k, t)| \right]. \end{aligned} \quad (3.3)$$

The phase-integral approximation solutions are of the order $2N + 1$, so $N = 2$ means that we have the approximation up to fifth order and $q_S(k, t)$ can be expanded in the form

$$q_S(k, t) = \sum_{n=0}^2 Y_{2n_S}(k, t) Q_S(k, t) = [Y_{0_S}(k, t) + Y_{2_S}(k, t) + Y_{4_S}(k, t)] Q_S(k, t). \quad (3.4)$$

In order to compute $q_S(k, t)$, we compute $Y_{2_S}(k, t)$, $Y_{4_S}(k, t)$, and the required functions $\varepsilon_{0_S}(k, t)$, $\varepsilon_{2_S}(k, t)$. The expression (3.4) gives a fifth-order approximation for $q_S(k, t)$. In

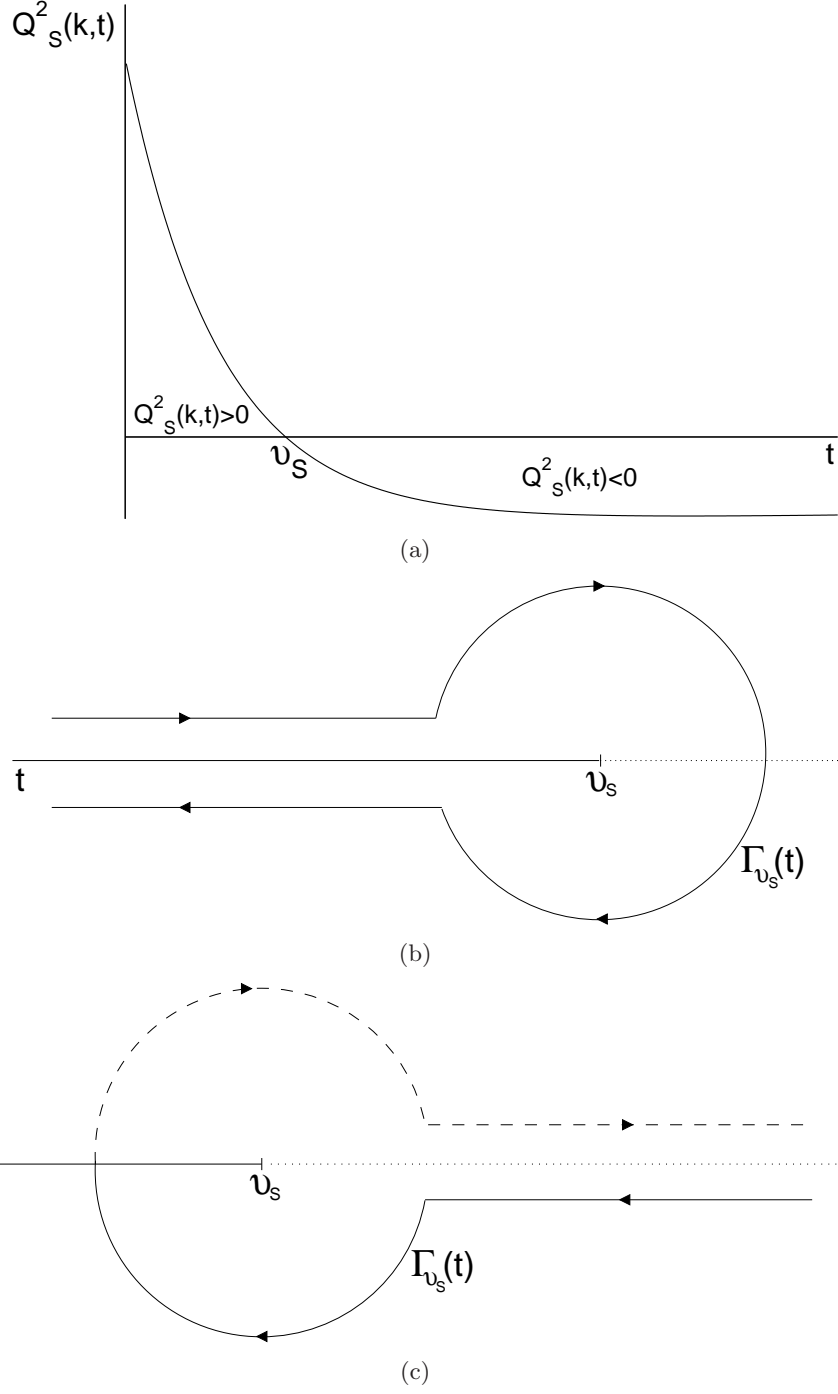


Figure 2. (a) Behavior of $Q_S^2(k, t)$. (b) Contour of integration $\Gamma_{\nu_S}(t)$ for $0 < t < \nu_S$. (c) Contour of integration $\Gamma_{\nu_S}(t)$ for $t > \nu_S$. The contour of integration has been done in the complex z plane. For this function we have a Riemann surface with two sheets where the function is single-valued. In the contour of integration the solid line indicates the first Riemann sheet whereas that the second Riemann sheet is indicated for the dashed line [29].

order to compute $\omega_S(k, t)$ we make a contour integration following the path indicated in Fig. 2(b)-(c).

$$\omega_S(k, t) = \omega_{0S}(k, t) + \sum_{n=1}^2 \omega_{2nS}(k, t), \quad (3.5)$$

$$= \int_{v_S}^t Q_S(k, t) dt + \frac{1}{2} \sum_{n=1}^2 \int_{\Gamma_{v_S}} Y_{2nS}(k, t) Q_S(k, t) dt, \quad (3.6)$$

$$= \int_{v_S}^t Q_S(k, t) dt + \frac{1}{2} \sum_{n=1}^2 \int_{\Gamma_{v_S}} f_{2nS}(k, t) dt, \quad (3.7)$$

where

$$f_{2nS}(k, t) = Y_{2nS}(k, t) Q_S(k, t). \quad (3.8)$$

The functions $f_{2nS}(k, t)$ have the following functional dependence:

$$f_{2S}(k, t) = A(k, t)(t - v_S)^{-5/2}, \quad (3.9)$$

$$f_{4S}(k, t) = B(k, t)(t - v_S)^{-11/2}, \quad (3.10)$$

$$(3.11)$$

where $A(k, t)$ and $B(k, t)$ are regular at v_S . With the help of the functions (3.9) we compute the integrals for ω_{2n} up to $N = 4$ using the contour indicated in Fig. 2(b)-(c). The expressions for ω_{2n} permit one to obtain the fifth-order phase integral approximation of the solution to the equation for scalar perturbations (2.9). The constants c_1 and c_2 are obtained using the limit $k t \rightarrow 0$ of the solutions on the left side of the turning point (3.2), and are given by the expressions

$$c_1 = -i c_2, \quad (3.12)$$

$$c_2 = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} e^{-i[k \eta(0) + |\omega_{0S}(k, 0)|]}. \quad (3.13)$$

In order to compute the scalar power spectrum, we need to calculate the limit as $k t \rightarrow \infty$ of the growing part of the solutions on the right side of the turning point given by Eq. (3.3) for scalar perturbations.

$$P_S(k) = \lim_{-kt \rightarrow \infty} \frac{k^3}{2\pi^2} \left| \frac{u_k^{\text{pi}}(t)}{z_S(t)} \right|^2. \quad (3.14)$$

3.2 Improved Uniform approximation

We want to obtain an approximate solution to the differential equation (2.9) in the range where $Q_S^2(k, t)$ have a simple root at $t_{\text{ret}} = v_S$, so that $Q_S^2(k, t) > 0$ for $0 < t < t_{\text{ret}}$ and $Q_S^2(k, t) < 0$ for $t > t_{\text{ret}}$ as depicted in Fig. 2(a). Using the uniform approximation method [1–3, 16, 30], we obtain that for $0 < t < t_{\text{ret}}$ we have

$$U_k(k, t) = \left[\frac{\rho_l(k, t)}{Q_S^2(k, t)} \right]^{1/4} \{C_1 A_i[-\rho_l(k, t)] + C_2 B_i[-\rho_l(k, t)]\}, \quad (3.15)$$

$$\frac{2}{3} [\rho_l(k, t)]^{3/2} = \int_t^{t_{\text{ret}}} [Q_S^2(k, t)]^{1/2} dt, \quad (3.16)$$

where C_1 and C_2 are two constants to be determined with the help of the boundary conditions (2.12). For $t > t_{\text{ret}}$

$$U_k(k, t) = \left[\frac{-\rho_r(k, t)}{Q_S^2(k, t)} \right]^{1/4} \{C_1 A_i[\rho_r(k, t)] + C_2 B_i[\rho_r(k, t)]\}, \quad (3.17)$$

$$\frac{2}{3} [\rho_r(k, t)]^{3/2} = \int_{t_{\text{ret}}}^t [-Q_S^2(k, t)]^{1/2} dt, \quad (3.18)$$

For the computation of the power spectrum we need to take the limit $kt \rightarrow \infty$ of the solution (3.17). In this limit we have

$$u_k^{\text{ua}}(t) \rightarrow \frac{C}{\sqrt{2a(t)}} [-Q_S^2(k, t)]^{-1/2} \left\{ \frac{1}{2} \exp \left(- \int_{v_S}^t [-Q_S^2(k, t)]^{1/2} dt \right) + i \exp \left(\int_{v_S}^t [-Q_S^2(k, t)]^{1/2} dt \right) \right\}, \quad (3.19)$$

where C is a phase factor. Notice that Eq. (3.19) is identical to Eq. (3.3) obtained in the first-order phase-integral approximation. Using Eqs. (2.6), (2.8) and the growing part of the solutions (3.19), one can compute the scalar and power spectrum using the uniform approximation method,

$$P_S(k) = \lim_{-kt \rightarrow \infty} \frac{k^3}{2\pi^2} \left| \frac{u_k^{\text{ua}}(t)}{z_S(t)} \right|^2. \quad (3.20)$$

We using the second-order improved uniform approximation for the power spectrum [18],

$$\tilde{P}_S(k) = P_S(k) [\Gamma^*(\bar{v}_S)], \quad (3.21)$$

where \bar{v}_S is the turning point for the scalar power spectrum and

$$\Gamma^*(v) \equiv 1 + \frac{1}{12v} + \frac{1}{288v^2} - \frac{139}{51840v^3} + \dots. \quad (3.22)$$

3.3 Slow-roll approximation

The scalar power spectra in the slow-roll approximation to second-order is given by the expression [13, 31]

$$P_S^{\text{sr}}(k) \simeq \left[1 + (4b - 2)\epsilon_1 + 2b\delta_1 + \left(4b^2 - 23 + \frac{7\pi^2}{3} \right) \epsilon_1^2 + \left(3b^2 + 2b - 22 + \frac{29\pi^2}{12} \right) \epsilon_1 \delta_1 + \left(3b^2 - 4 + \frac{5\pi^2}{12} \right) \delta_1^2 + \left(-b^2 + \frac{\pi^2}{12} \right) \delta_2 \right] \left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}} \right)^2 \Big|_{k=aH}, \quad (3.23)$$

where $b \simeq 0.7296$ is the Euler constant. The spectral index in the slow-roll approximation to second-order is

$$n_S^{\text{sr}}(k) \simeq 1 - 4\epsilon_1 - 2\delta_1 + (8c - 8)\epsilon_1^2 + (10c - 6)\epsilon_1\delta_1. \quad (3.24)$$

The parameters used in the equations (3.23) and (3.24), express in terms of the potential $V(\phi)$, are given by [13]

$$H^2 \simeq \frac{V}{3} \left(1 + \frac{1}{6}U_1 - \frac{1}{12}U_1^2 + \frac{1}{9}U_1V_1 \right), \quad (3.25)$$

$$\left(\frac{H}{\dot{\phi}} \right)^2 \simeq \frac{1}{U_1} \left(1 + \frac{2}{3}U_1 - \frac{2}{3}V_1 - \frac{4}{9}U_1^2 + \frac{7}{9}U_1V_1 - \frac{1}{9}V_1^2 - \frac{2}{9}V_2 \right), \quad (3.26)$$

$$\epsilon_1 \simeq \frac{1}{2}U_1 - \frac{1}{3}U_1^2 + \frac{1}{3}U_1V_1 + \frac{4}{9}U_1^3 - \frac{5}{6}U_1^2V_1 + \frac{5}{18}U_1V_1^2 + \frac{1}{9}U_1V_2, \quad (3.27)$$

$$\delta_1 \simeq \frac{1}{2}U_1 - V_1 - \frac{2}{3}U_1^2 + \frac{4}{3}U_1V_1 - \frac{1}{3}V_1^2 - \frac{1}{3}V_2 + \frac{3}{2}U_1^3 - 4U_1^2V_1 + \frac{23}{9}U_1V_1^2 - \frac{2}{9}V_1^3 + \frac{17}{18}U_1V_2 - \frac{2}{3}V_1V_2 - \frac{1}{9}V_3, \quad (3.28)$$

$$\delta_2 \simeq U_1^2 - \frac{5}{2}U_1V_1 + V_1^2 + V_2 - \frac{19}{6}U_1^3 + \frac{55}{6}U_1^2V_1 - \frac{13}{2}U_1V_1^2 + \frac{2}{3}V_1^3 - \frac{5}{2}U_1V_2 + 2V_1V_2 + \frac{1}{3}V_3. \quad (3.29)$$

With,

$$U_1 \equiv \left(\frac{V'}{V} \right)^2, \quad (3.30)$$

$$V_1 \equiv \frac{V''}{V}, \quad (3.31)$$

$$V_2 \equiv \frac{V'V'''}{V^2}, \quad (3.32)$$

$$V_3 \equiv \frac{(V')^2 V''''}{V^3}. \quad (3.33)$$

The expressions (3.25)-(3.33) depend explicitly on ϕ . In order to compute the scalar power spectrum we need to obtain the dependence on the variable k . For a given value of k ($0.0001 \text{ Mpc}^{-1} \leq k \leq 15 \text{ Mpc}^{-1}$) we obtain t from the relation $k = aH$. Thus, for each k one obtains a value of t that we substitute into Eq. (2.4).

3.4 Numerical solution

We integrate on the physical time t the equations (2.9) governing the scalar perturbations using the predictor-corrector Adams method of order 12 [26], and solve two differential equations, one for the real part and another for the imaginary part U_k . Two initial conditions are needed in each case $U_k(t_i)$, $U'_k(t_i)$, which can be obtained from the third-order phase-integral approximation. We start the numerical integration at t_i calculated at 25 oscillations before

reaching the turning point t_{ret} [32]. We call this procedure ICs phi3. Fig. 3 compare the numerical solution with the fifth-order phase-integral approximation for $\text{Re}(u_k)$. Figures are plotted against the number of e-folds N . The solid line corresponds to the numerical solution (ICs phi3), the dashed line corresponds to the fifth-order phase-integral approximation. The turning point v_S is indicated with an arrow. We stop the numerical computation of $P_S(k)$ at $t = 5.00 \times 10^5 M_{\text{Pl}}^{-1}$, after the mode leaves the horizon, where u_k/z_S is approximately constant. Notice that the expressions for fitting (2.6) and (2.7) are valid in the aforementioned time scales, therefore we can use them for computing the scalar $P_S(k)$ power spectrum.

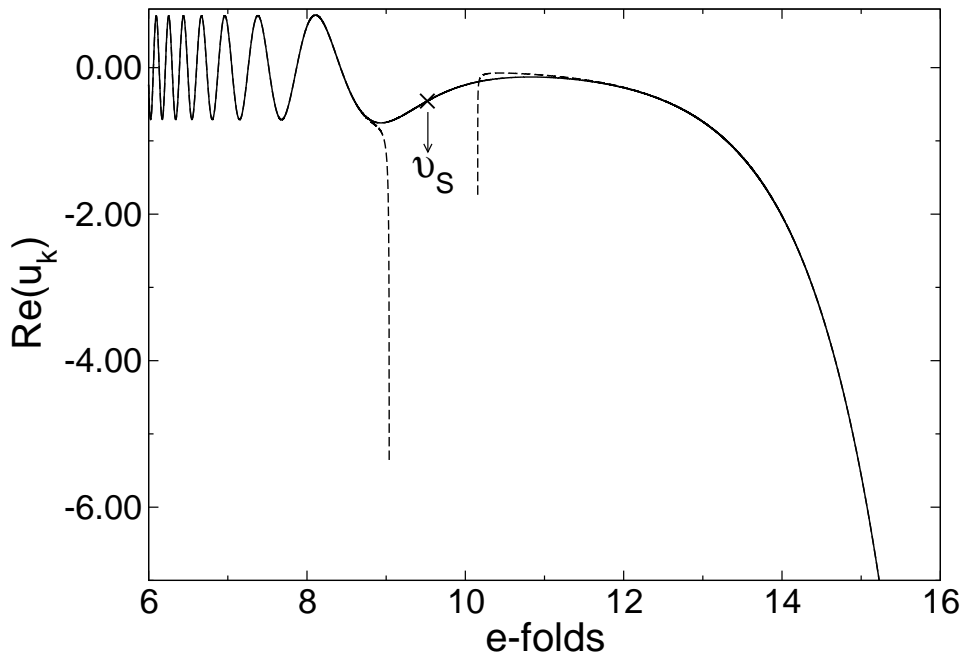


Figure 3. $\text{Re}(u_k)$ versus the number of e-folds for the chaotic inflationary $\frac{1}{4}\lambda\phi^4$ model. Solid line: numerical result (ICs phi3); dashed line: fifth-order phase-integral approximation.

4 Results

For the chaotic $\frac{1}{4}\lambda\phi^4$ inflationary model, we want to compare the scalar power spectra for different values of k calculated using the third and fifth-order phase-integral approximation with the numerical result (ICs phi3), the first and second-order slow-roll approximation and the first and second-order uniform approximation method. First we analyze the results for the scalar $P_S(k)$ power spectrum shown in Fig. 4.

Table 1 shows the value of $P_S(k)$ and the relative error $P_S(k)$ using each method of approximation at the WMAP pivot scale [18]. It can be observed that the best value is obtained with the fifth-order phase-integral approximation. It should be noticed that the slow-roll approximation works well since the parameters ϵ_1 , ϵ_2 and δ_n are small.

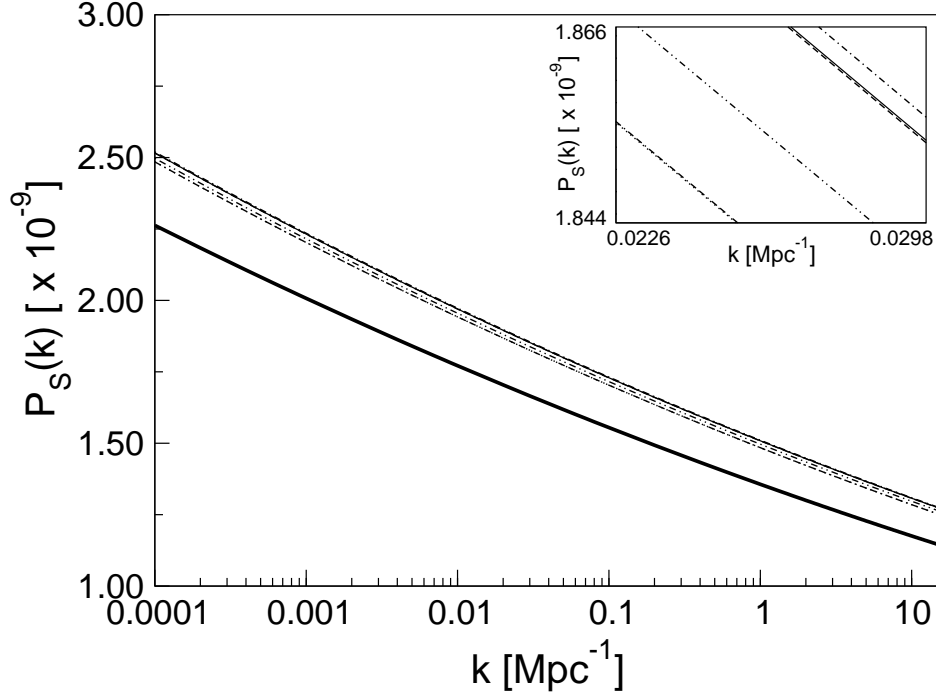


Figure 4. $P_S(k)$ for the chaotic inflationary $\frac{1}{4}\lambda\phi^4$ model. Thin solid line: numerical result (ICs phi3); dot-dashed line: third-order phase-integral approximation; dashed line: fifth-order phase-integral approximation; thick solid line: first-order phase-integral approximation, WKB and first-order uniform approximation; dashed double-dots line: second-order improved uniform approximation, double-dashed dot line: second-order slow-roll approximation; dotted line: first-order slow-roll approximation. The inset is an enlargement of the figure.

Method	$P_S(k) \times 10^{-9}$	rel. error $P_S(k)$
Numerical	1.7995	
Third-order phase-integral approximation	1.8021	0.1407 %
Fifth-order phase-integral approximation	1.7993	0.0149 %
Leading order of slow-roll approximation	1.7889	0.5866 %
First-order slow-roll approximation	1.7952	0.2394 %
Second-order slow-roll approximation	1.7953	0.2324 %
First-order phase integral approximation, WKB, and first-order uniform approximation	1.6181	10.0851 %
Second-order improved uniform approximation	1.7857	0.7694 %

Table 1. Value of $P_S(k)$ and its relative error obtained with different approximation methods for the chaotic inflationary model $\frac{1}{4}\lambda\phi^4$ for the mode $k = 0.05 \text{ Mpc}^{-1}$.

The first-order phase integral approximation, the WKB and the first-order uniform approximation give the same result, and deviate from the numerical result in 10.0851 %.

The second-order-improved uniform approximation gives an error of 0.7694 %. With the leading, first and second-order slow-roll approximation we have an error for $P_S(k)$ of $\gg 100\%$, 0.2394 % and 0.2324 %, respectively. Using the third-order phase-integral approximation the error gives 0.1407 %, whereas the fifth-order phase-integral, due to the convergence of the method, reduces to 0.0149 % for $P_S(k)$. Note that rel. error $P_S(k)$ with sr1 \approx rel. error $P_S(k)$ with sr2, which is unexpected according to the results obtained previously for the power-law inflationary model [17]. The computation time is different for each method, for the scalar power spectrum we have the following times: numerical \approx several hours, pi3 \approx few minutes, pi5 \approx several minutes, sr0 \approx few seconds, sr1 \approx few seconds, sr2 \approx few seconds, WKB, ua1, pi1 \approx few seconds, ua2 \approx few seconds. Fig. 5 shows the results for the spectral indice $n_S(k)$.

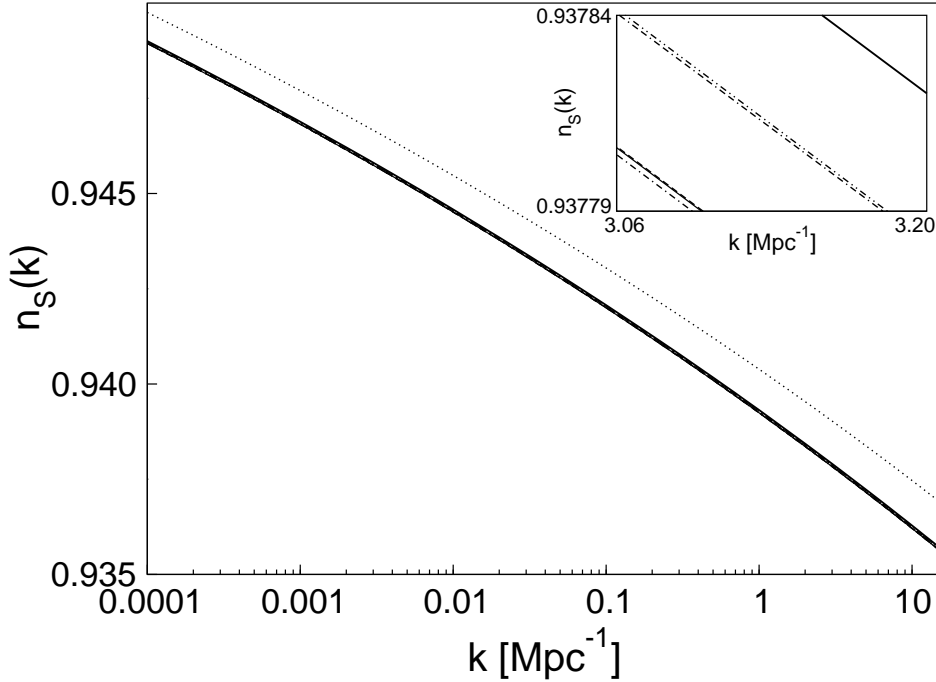


Figure 5. $n_S(k)$ for the chaotic inflationary $\frac{1}{4}\lambda\phi^4$ model. Thin solid line: numerical result (ICs phi3) and fifth-order phase-integral approximation; dot-dashed line: third-order phase-integral approximation; thick solid line: first-order phase-integral approximation, WKB and first-order uniform approximation; dashed double-dots line: second-order improved uniform approximation; double-dashed dot line: second-order slow-roll approximation; dotted line: first-order slow-roll approximation. The inset is an enlargement of the figure.

Table 2 shows the value of $n_S(k)$ and the relative error $n_S(k)$ using each method of approximation at the WMAP pivot scale [18]. The first-order phase integral approximation, the WKB and the first-order uniform approximation give the same result, and deviate from the numerical result in 0.007239 %. The second-order-improved uniform approximation gives an error of 0.003909 %. With the leading, first and second-order slow-roll approximation we have an error for $n_S(k)$ of 6.0692239 %, 0.116637 % and 0.0012000 %, respectively. Using the third-order phase-integral approximation the error gives 0.000157 %, whereas the fifth-order

phase-integral, due to the convergence of the method, reduces to 0.000026 % for $n_S(k)$. Note that the first and second-order slow-roll approximation gives a better results for the spectral indice $n_S(k)$ than the power spectrum $P_S(k)$.

Method	$n_S(k)$	rel. error $n_S(k)$
Numerical	0.942780	
Third-order phase-integral approximation	0.942779	0.000157 %
Fifth-order phase-integral approximation	0.942781	0.000026 %
Leading order of slow-roll approximation	1	6.069239 %
First-order slow-roll approximation	0.943880	0.116637 %
Second-order slow-roll approximation	0.942894	0.012000 %
First-order phase integral approximation, WKB, and first-order uniform approximation	0.942849	0.007239 %
Second-order improved uniform approximation	0.942817	0.003909 %

Table 2. Value of $n_S(k)$ and its relative error obtained with different approximation methods for the chaotic inflationary model $\frac{1}{4}\lambda\phi^4$ for the mode $k = 0.05 \text{ Mpc}^{-1}$.

The relative errors with respect to the numerical result have been obtained using the expressions:

$$\text{rel. error } P_S(k) = \frac{[P_S^{\text{approx}}(k) - P_S^{\text{num}}(k)]}{P_S^{\text{num}}(k)} \times 100, \quad (4.1)$$

$$\text{rel. error } n_S(k) = \frac{[n_S^{\text{approx}}(k) - n_S^{\text{num}}(k)]}{n_S^{\text{num}}(k)} \times 100. \quad (4.2)$$

5 Conclusion

As in our previous work [1–3], we show that the phase-integral approximation is an efficient method to calculate the scalar (or tensor) power spectrum and the spectral index in several models of inflation. Our results indicate that the scalar power spectrum is almost scale invariant since we have considered the usual scenario where the observable modes cross the horizon before the inflation ends, unlike the proposal presented by Ramirez *et. al* [8]. Therefore, the good agreement between the numerical results and those obtained with the phase-integral approximation shows that the phase integral method is a very useful approximation tool for computing the scalar and tensor the power spectra in a wide range of inflationar scenarios.

References

- [1] C. Rojas and V. M. Villalba, *Computation of inflationary cosmological perturbations in the power-law inflationary model using the phase-integral method*, *Phys. Rev. D* **75** (2007) 063518.
- [2] V. M. Villalba and C. Rojas, *Applications of the phase integral method ins ome inflationary scenarios*, *J. Phys. Conf. Ser.* **66** (2007) 012034.
- [3] C. Rojas and V. M. Villalba, *Computation of inflationary cosmological perturbations in chaotic inflationary scenarios using the phase-integral method*, *Phys. Rev. D* **79** (2009) 103502.

- [4] A. D. Linde, *Chaotic inflating universe*, *JETP Lett.* **38** (1983) 176.
- [5] A. D. Linde, *Chaotic inflation*, *Phys. Lett. B* **129** (1983) 177.
- [6] E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennet, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, *Seven-year wilkinson microwave anisotropy probe (wmap?) observations: Cosmological interpretation*, *Astroph* **192** (2011) 18.
- [7] E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennet, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, S. S. Meyer, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, *Five-year wilkinson microwave anisotropy probe observations: cosmological interpretation*, *Astrophys. J* **180** (2009) 330.
- [8] E. Ramirez and D. J. Schwarz, ϕ^4 inflation is not excluded, *Phys. Rev. D* **80** (2009) 023525.
- [9] E. D. Stewart and D. H. Lyth, *A more accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation*, *Phys. Lett. B* **302** (1993) 171.
- [10] R. E. Langer, *On the connection formulas and the solutions of the wave equation*, *Phys. Rev.* **51** (1937) 669.
- [11] J. Martin and D. J. Schwarz, *Wkb approximation for inflationary cosmological perturbations*, *Phys. Rev. D* **67** (2003) 083512.
- [12] R. Casadio, F. Finelli, M. Luzzi, and G. Venturi, *Improved wkb analysis of cosmological perturbations*, *Phys. Rev. D* **71** (2005) 043517.
- [13] E. D. Stewart and J. Gong, *The density perturbation power spectrum to second-order corrections in the slow-roll expansion*, *Phys. Lett. B* **510** (2001) 1.
- [14] R. Casadio, F. Finelli, M. Luzzi, and G. Venturi, *Higher order slow-roll predictions for inflation*, *Phys. Lett. B* **625** (2005) 1.
- [15] R. Casadio, F. Finelli, M. Luzzi, and G. Venturi, *Improved wkb analysis of slow-roll inflation*, *Phys. Rev. D* **72** (2005) 103516.
- [16] S. Habib, A. Heinen, K. Heitmann, G. Jungman, and C. Molina-París, *The inflationary perturbation spectrum*, *Phys. Rev. Lett.* **89** (2002) 281301.
- [17] S. Habib, A. Heinen, K. Heitmann, G. Jungman, and C. Molina-París, *Characterizing inflationary perturbations: The uniform approximation*, *Phys. Rev. D* **70** (2004) 083507.
- [18] S. Habib, A. Heinen, K. Heitmann, and G. Jungman, *Inflationary perturbations and precision cosmology*, *Phys. Rev. D* **71** (2005) 043518.
- [19] R. Casadio, F. Finelli, A. Kamenshchik, M. Luzzi, and G. Venturi, *The method of comparison equations for cosmological perturbations*, *J. Cosmol. Astropart. Phys.* **0604** (2006) 1.
- [20] R. B. Dingle, *The method of comparison equations in the solution of linear second-order differential equations (generalized w.k.b. method)*, *Appl. Sci. Res. B.* **5** (1956) 345.
- [21] S. V. Miller and R. H. Good, *A wkb-type approximation to the schrödinger equation*, *Phys. Rev.* **91** (1953) 174.
- [22] M. Berry, *Waves near stokes lines*, *Proc. R. Soc. Lond. A* **427** (1990) 265.
- [23] D. Larson, J. Dunkley, G. Hinshaw, E. Komatsu, M. R. Nolta, C. L. Bennett, B. Gold, M. Halpern, R. S. Hill, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, L. Page, K. M. Smith, D. N. Spergel, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, *Seven-year wilkinson microwave anisotropy probe (wmap) observations. power spectra and wmap-derived parameters*, *Astrophys. J* **192** (2011) 16.
- [24] L. Alabidi and I. Huston, *An update on single field models of inflation in light of wmap7*, *J.*

Cosmol. Astropart. Phys. **08** (2010) 037.

- [25] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*. Cambridge University Press, 2000.
- [26] V. F. Gerald and P. O. Wheatley., *Applied Numerical Analysis*. Addison-Wesley Pub. Co., 1984.
- [27] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover, New York, 1965.
- [28] N. Fröman and P. O. Föman, *Phase-integral formulas for Bessel functions and their relation to already existing asymptotic formulas*, vol. 40. Springer Tracts in Natural Philosophy, 1996.
- [29] P. Froman, F. Karlsson, and S. Yngve, *Phase-Integral Method. Allowing Nearlying Transition Points*, *J. Math. Phys.* **27** (1986) 2738.
- [30] M. Berry and K. E. MounT, *Semiclassical approximations in wave mechanics*, *Rep. Prog. Phys.* **35** (1972) 315.
- [31] J. O. Gong, *General slow-roll spectrum for gravitational waves*, *Class. Quantum Grav.* **21** (2004) 5555.
- [32] V. E. Cunha, *Evolution of background and perturbation equations in single field inflation models*, tech. rep., Universtiy of Chicago, 2005.